

## Noetherian rings

An important property in commutative algebra is finite generation of ideals.

For a polynomial ring in finitely many variables over a field  $k[x_1, \dots, x_n]$ , the fact (which we will show) that every ideal is finitely generated is equivalent to the fact that every variety in  $A^n$  is the intersection of finitely many hypersurfaces (i.e. the zero locus of a single polynomial).

**Def:** A ring  $R$  is Noetherian if every ideal of  $R$  is finitely generated.

Equivalently,  $R$  is Noetherian if its ideals satisfy the ascending chain condition:

**Claim:**  $R$  is Noetherian iff every strictly increasing chain of ideals terminates.

**Pf:** If  $I$  is not finitely generated, choose  $f_1 \in I$ ,  $f_2 \in I \setminus (f_1)$ ,  $f_n \in I \setminus (f_1, \dots, f_{n-1})$ . Then we get an infinite chain of ideals

$$(f_1) \subseteq (f_1, f_2) \subseteq \dots$$

Conversely, if  $I_1 \subsetneq I_2 \subsetneq \dots$  is a chain of ideals and  $I = \bigcup I_i$  is finitely generated, then all of the generators are in one of the  $I_j$ , so  $I = I_j$ .  $\square$

**Ex:** Some familiar Noetherian rings: all fields,  $\mathbb{Z}$ ,  $\mathbb{Z}[x]$

**Thm:** (Hilbert Basis Theorem) If  $R$  is a Noetherian ring, then  $R[x]$  is Noetherian.

(Note that this implies  $R[x_1, \dots, x_n]$  is also Noetherian by induction.)

**Pf:** First, some terminology: If  $f = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \in R[x]$ , w/  $a_n \neq 0$ , then  $a_n$  is the initial coefficient,  $a_n x^n$  is the initial term.

Let  $I \subseteq R[x]$  be an ideal. Choose a sequence  $f_1, f_2, \dots \in I$  as follows: let  $f_1 \in I$  be a nonzero elt of least degree,

and let  $f_{n+1}$  be an element of least degree in  $I \setminus (f_1, \dots, f_n)$

If  $(f_1, \dots, f_n) = I$ , we're done.

Let  $a_j$  be the initial coefficient of  $f_j$ . Then  $J = (a_1, a_2, \dots) \subseteq R$  is finitely generated. Let  $m$  be the smallest integer s.t.  $J = (a_1, \dots, a_m)$ .

Claim:  $I = (f_1, \dots, f_m)$ .

Otherwise consider  $f_{m+1}$ . Then  $a_{m+1} = \sum_{j=1}^m u_j a_j$  for some  $u_j \in R$ .

Since  $\deg f_{m+1} \geq \deg f_j$  for  $j \leq m$ , we can define

$$g = \sum_{j=1}^m u_j f_j x^{\deg f_{m+1} - \deg f_j} \in (f_1, \dots, f_m).$$

$f_{m+1} - g \in I \setminus (f_1, \dots, f_m)$  and has degree strictly less than the degree of  $f_{m+1}$ , which is a contradiction.  $\square$

Cor: If  $R$  is Noetherian, and  $S$  a finitely generated  $R$ -algebra, then  $S$  is Noetherian.

Pf:  $S = R[a_1, \dots, a_n]$ , so  $R[x_1, \dots, x_n]$ , which is Noetherian, surjects onto  $S$ . Thus, any  $I$  in  $S$  is generated by the images of generators of its preimage.  $\square$

## Noetherian Modules

An  $R$ -module  $M$  is Noetherian if every submodule is finitely generated. (Equivalently, if it has the ascending chain condition on submodules or if every collection of submodules has a maximal element — see Hw.)

Prop: If  $R$  is a Noetherian ring and  $M$  a finitely generated  $R$ -module, then  $M$  is Noetherian.

Pf: Let  $f_1, \dots, f_n$  be the generators of  $M$  and  $N \subseteq M$  a submodule.

If  $n=1$ , consider the map  $R \rightarrow M$  sending  $1 \mapsto f_1$ .  
Then the preimage of  $N$  is an ideal, so the images of its generators generate  $N$ .

If  $n > 1$ , by induction,  $M/Rf_1$  is Noetherian.  
So if  $\bar{N}$  is the image of  $N$  in the quotient,  
 $\bar{N}$  is f.g. by the images of  $g_1, \dots, g_s$ .

$N \cap Rf_1$  is a submodule of  $Rf_1$ , so it's f.g. by  $h_1, \dots, h_r$ .

Thus, if  $a \in N$ ,  $\bar{a}$  is a lin comb. of the  $g_i$ , so  $a$  is gen by the  $g_i$  and  $h_j$ .  $\square$